Design Criteria for Predicting Damping in Underdamped Linear Lumped-Parameter Systems

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Simple design criteria that determine whether the modal damping ratios of an underdamped linear lumped-parameter system are greater than, equal to, or less than a given non-negative real scalar are presented. The modal damping ratio expression is an extension of the usual single degree-of-freedom relation. The damping ratio conditions are derived from the modal damping ratio expression and the definiteness of certain combinations of coefficient matrices. An example illustrates how the damping ratio prediction relations are used in design.

I. Introduction

QUANTIFYING, characterizing, and/or controlling the rate of energy loss or decay rate of a vibrating structure is a frequent topic of discussion in the vibration literature. One common measure of the vibration decay rate of a system is the system's modal damping ratios. For systems with many degrees of freedom and/or non-normal modes, the modal damping ratios are difficult to obtain. Rather than calculating the modal damping ratios, a designer would prefer to gain some insight as to the values of the modal damping ratios directly from the system parameters. Here, criteria are derived which are written in terms of the system parameters and which when satisfied determine whether the modal damping ratios of the system are greater than, equal to, or less than a given non-negative real scalar. The design criteria are valid if the system is stable and underdamped. Lastly, the use of the modal damping ratio prediction criteria in design is illustrated with an example.

II. Modal Damping Ratio Prediction Criteria

In this paper, several modal damping ratio prediction criteria are presented. First, a general analytical expression for the modal damping ratios of a system is derived. The analytical expression is valid for stable, underdamped systems. It is shown that this modal damping ratio expression is an extension of the usual single degree-of-freedom (SDOF) relation. Afterwards, each modal damping ratio prediction criterion is derived in turn.

A. Analytical Expression for System Modal Damping Ratios

Consider a multiple degree-of-freedom (MDOF) linear system defined by the matrix-vector differential equation

$$M\ddot{x} + C\dot{x} + Kx = f(t) \tag{1}$$

where M, C, and K are real and symmetric $n \times n$ mass, damping, and stiffness matrices, respectively; $x(t)$ is an n-dimensional vector of physical coordinates; the overdot denotes the first time derivative; and $f(t)$ is an n-dimensional vector of forcing functions. To obtain a stable response, it is assumed that the matrices M, C, and K in Eq. (1) are positive definite, positive semidefinite, and positive definite, respectively.[1,2] In addition, the system defined by Eq. (1) is underdamped (i.e., each system eigenvalue appears as a complex conjugate pair with negative real part) if the matrix $(2\tilde{K}^{1/2} - \tilde{C})$ is positive definite,[3] where $\tilde{C} = M^{-1/2}CM^{-1/2}$, $\tilde{K} = M^{-1/2}KM^{-1/2}$, and $M^{-1/2}$ is the inverse of the positive definite square root of the matrix M.

As an aside, the underdamped condition can be re-expressed in two alternate forms. Since the matrix $4\tilde{K}$ is positive definite and the matrix $\tilde{C}$ is positive semidefinite, then according to Bellman,[4] if the matrix $(4\tilde{K} - \tilde{C}^2)$ is positive definite so is the matrix $(2\tilde{K}^{1/2} - \tilde{C})$. Therefore, if the matrix $(4\tilde{K} - \tilde{C}^2)$ is positive definite then the system is underdamped. This underdamped condition avoids computation of the square root of $\tilde{K}$; therefore, this condition is simpler to evaluate than the underdamped condition stated in the previous paragraph.

Now, another simpler underdamped condition is derived. If $(4\tilde{K} - \tilde{C}^2)$ is positive definite, then

$$y^*(4\tilde{K} - \tilde{C}^2)y > 0 \tag{2}$$

for all complex n-dimensional vectors y (Ref. 5). Substituting $\tilde{C} = M^{-1/2}CM^{-1/2}$ and $\tilde{K} = M^{-1/2}KM^{-1/2}$ into Eq. (2) and simplifying results in

$$z^*(4K - CM^{-1}C)z > 0 \tag{3}$$

where $z = M^{-1/2}y$. Since $M^{-1/2}$ is positive definite, then z is a nonzero complex n-dimensional vector; therefore, Eq. (3) is a statement that $(4K - CM^{-1}C)$ is positive definite. Consequently, since Eqs. (2) and (3) are equivalent then $(4K - CM^{-1}C)$ is positive definite if and only if $(4\tilde{K} - \tilde{C}^2)$ is positive definite. Therefore, if the matrix $(4K - CM^{-1}C)$ is positive definite then the system is underdamped. Note that since the evaluation of the square root of M is avoided, then this underdamped condition requires less computation than the previous two. In summary, the system given by Eq. (1) is underdamped if at least one of the matrices $(2\tilde{K}^{1/2} - \tilde{C})$, $(4\tilde{K} - \tilde{C}^2)$, or $(4K - CM^{-1}C)$ is positive definite.

Continuing with the modal damping ratio derivation, the system eigenvalues and eigenvectors are obtained from the homogeneous form of Eq. (1), or

$$M\ddot{x} + C\dot{x} + Kx = \mathbf{0} \tag{4}$$

where $\mathbf{0}$ is an n-dimensional vector of zeros. Assume a solution to Eq. (4) of the form $x(t) = ve^{\lambda t}$, where λ is a complex scalar

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and v is a complex n-dimensional vector. Substituting into Eq. (4) and simplifying yields the system eigenvector-value problem

$$(M\lambda^2+C\lambda+K)v = 0 \tag{5}$$

Solving Eq. (5) for λ and v yields $2n$ system eigenvalues, $\{\lambda_p, p=1,2,\ldots,2n\}$, and $2n$ corresponding system eigenvectors, $\{v_p, p=1,2,\ldots,2n\}$. In the underdamped case, the system eigenvalues and eigenvectors exist in complex conjugate pairs.[3] Reorganizing yields n pairs of complex conjugate system eigenvalues, $\{\lambda_j,\bar\lambda_j, j=1,2,\ldots,n\}$, and corresponding system eigenvectors, $\{v_j,\bar v_j, j=1,2,\ldots,n\}$, where the overbar denotes the complex conjugate.

Consider the jth complex conjugate pair of system eigenvalues, λ_j and $\bar\lambda_j$, and corresponding eigenvectors, v_j and $\bar v_j$. Equation (5) is valid for either element of the complex conjugate pair, or

$$(M\lambda_j^2+C\lambda_j+K)v_j = 0 \tag{6}$$

$$(M\bar\lambda_j^2+C\bar\lambda_j+K)\bar v_j = 0 \tag{7}$$

Premultiplying Eq. (6) by v_j^* and Eq. (7) by $\bar v_j^*$, and solving for λ_j and $\bar\lambda_j$, respectively, yields

$$\lambda_j,\alpha = (-c_j/2m_j) \pm (\sqrt{c_j^2-4m_jk_j}/2m_j) \tag{8}$$

$$\beta,\bar\lambda_j = (-\bar c_j/2\bar m_j) \pm (\sqrt{\bar c_j^2-4\bar m_j\bar k_j}/2\bar m_j) \tag{9}$$

where $m_j=v_j^*Mv_j$, $c_j=v_j^*Cv_j$, $k_j=v_j^*Kv_j$, $\bar m_j=\bar v_j^*M\bar v_j$, $\bar c_j=\bar v_j^*C\bar v_j$, $\bar k_j=\bar v_j^*K\bar v_j$, and * denotes the complex conjugate transpose. Examining Eqs. (8) and (9), it is implied that $\alpha=\bar\lambda_j$ and $\beta=\lambda_j$. This statement, however, requires a proof.

Consider a symmetric and real $n\times n$ matrix A and a complex n-dimensional vector q. According to Lancaster[5]

$$q^*Aq = q_R^TAq_R + q_I^TAq_I \tag{10}$$

where the subscripts R and I denote the real and imaginary parts, respectively; and superscript T denotes the transpose. Replacing q with $\bar q$, the complex conjugate of q, in Eq. (10) results in

$$\bar q^*A\bar q = \bar q_R^TA\bar q_R + \bar q_I^TA\bar q_I \tag{11}$$

Since $\bar q_R=q_R$ and $\bar q=-q_I$, then Eq. (11) becomes

$$\bar q^*A\bar q = q_R^TAq_R + q_i^TAq_I \tag{12}$$

Comparing Eqs. (10) and (12), the right-hand sides are equivalent; therefore,

$$q^*Aq = \bar q^*A\bar q \tag{13}$$

Given that M, C, and K are real and symmetric matrices, applying Eq. (13) to Eqs. (8) and (9) yields that $m_j=\bar m_j$, $c_j=\bar c_j$, and $k_j=\bar k_j$. Therefore, the right-hand sides of Eqs. (8) and (9) are equivalent. Consequently, $\alpha=\bar\lambda_j$ and $\beta=\lambda_j$, and

$$\lambda_j,\bar\lambda_j = (-c_j/2m_j)\pm(\sqrt{c_j^2-4m_jk_j}/2m_j) \tag{14}$$

The alternate equivalent expression for the eigenvalue pair, λ_j and $\bar\lambda_j$, is obtained by replacing m_j, c_j, and k_j with $\bar m_j$, $\bar c_j$, and $\bar k_j$, respectively, in Eq. (14).

Next the imaginary and real parts of λ_j and $\bar\lambda_j$ are determined. Since the matrices M and K are real, symmetric, and positive definite, then the scalars m_j and k_j are real and positive.[6] Likewise, since the matrix C is real, symmetric, and positive semidefinite, then the scalar c_j is real and non-negative. Therefore, the term $-c_j/(2m_j)$ in Eq. (14) is real and nonpositive. In addition, the discriminant $(c_j^2-4m_jk_j)$ is real. The discriminant is also negative, since the system given by Eq. (1) was defined as underdamped.[3] Consequently, the term $\sqrt{c_j^2-4m_jk_j}/(2m_j)$ is purely imaginary. Therefore, the real and imaginary parts of λ_j and $\bar\lambda_j$ are given by

$$\mathrm{Re}(\lambda_j,\bar\lambda_j) = (-c_j/2m_j)$$

$$\mathrm{Im}(\lambda_j,\bar\lambda_j) = (\sqrt{4m_jk_j-c_j^2}/2m_j) \tag{15}$$

respectively.

Assume that the jth eigenvalue pair, λ_j and $\bar\lambda_j$, can be expressed in the form

$$\lambda_j,\bar\lambda_j = -\zeta_j\omega_j \pm i\omega_j\sqrt{1-\zeta_j^2}$$

where ζ_j is the jth modal damping ratio, ω_j is the jth natural frequency, and $i=\sqrt{-1}$. The jth natural frequency is real and non-negative. In addition, the jth modal damping ratio is real and bounded by $0\le\zeta_j<1$. Therefore,

$$\mathrm{Re}(\lambda_j,\bar\lambda_j) = -\zeta_j\omega_j, \qquad \mathrm{Im}(\lambda_j,\bar\lambda_j)=\omega_j\sqrt{1-\zeta_j^2} \tag{16}$$

Equating Eqs. (15) and (16), and solving for ζ_j in terms of m_j, c_j, and k_j yields

$$\zeta_j = (c_j/2\sqrt{m_jk_j}) \tag{17}$$

which is an expression for the jth modal damping ratio of the system. If the system given by Eq. (1) is underdamped, then Eq. (17) is valid.

To simplify later proofs and derivations, it is necessary to convert Eq. (17) to a more convenient form. Replacing m_j, c_j, and k_j with $v_j^*Mv_j$, $v_j^*Cv_j$, and $v_j^*Kv_j$, respectively, in Eq. (17) results in

$$\zeta_j = \left(v_j^*Cv_j/2\sqrt{v_j^*Mv_jv_j^*Kv_j}\right) \tag{18}$$

Rewriting Eq. (18) in terms of $\tilde C=M^{-1/2}CM^{-1/2}$ and $\tilde K=M^{-1/2}KM^{-1/2}$ yields

$$\zeta_j = (s_j^*\tilde Cs_j/2\sqrt{s_j^*s_js_j^*\tilde Ks_j}) \tag{19}$$

where $s_j=M^{1/2}v_j$. Equation (19) is the general expression for the jth modal damping ratio of an underdamped MDOF linear system.

In the SDOF case, since $n=1$ there is only one damping ratio associated with the system. Equation (19), therefore, reduces to

$$\zeta = (s^*\tilde cs/2\sqrt{s^*ss^*\tilde ks}) \tag{20}$$

where s, $\tilde c$, and $\tilde k$ are now scalars. Substituting $\tilde c=m^{-1/2}cm^{-1/2}$ and $\tilde k=m^{-1/2}km^{-1/2}$ into Eq. (20) and simplifying yields

$$\zeta = (c/2\sqrt{km})$$

which is the usual expression for the damping ratio of a SDOF system.[1]

B. "Upper Bound" Modal Damping Ratio Criteria

In this section three upper bound modal damping ratio criteria are derived. First, it is shown that if the matrix $(2a\tilde K^{1/2}-\tilde C)$ is positive semidefinite, then all $\zeta_j\le a$ where $a\ge 0$. If this statement holds, then the scalar a provides an upper bound on all of the system modal damping ratios. Afterwards, two alternate upper bound modal damping ratio criteria are presented.

By hypothesis, the matrix $(2a\tilde K^{1/2}-\tilde C)$ is positive semidefinite, where a is a real non-negative constant scalar. Since the matrices M, C, and K are real and symmetric, then the matrices $\tilde C$, $\tilde K$, and $(2a\tilde K^{1/2}-\tilde C)$ are also real and symmetric.

Therefore,

$$y^*(2a\tilde{K}^{1/2} - \tilde{C})y \geq 0 \tag{21}$$

for all complex n-dimensional vectors y (Ref. 5). Expanding Eq. (21) yields

$$2ay^*\tilde{K}^{1/2}y - y^*\tilde{C}y \geq 0 \tag{22}$$

The Cauchy-Schwarz inequality with norm and inner product defined by the usual scalar product of two vectors is

$$(q^*r)^2 \leq q^*qr^*r \tag{23}$$

where q and r are complex n-dimensional vectors.[7] Substituting $q = y$ and $r = \tilde{K}^{1/2}y$ into Eq. (23) results in

$$(y^*\tilde{K}^{1/2}y)^2 \leq y^*yy^*\tilde{K}y \tag{24}$$

Since K and M are positive definite, then $\tilde{K}$ is positive definite; therefore, both sides of Eq. (24) are positive real scalars. Consequently, taking the positive square root of both sides of Eq. (24) will not change the nature of the inequality, or

$$y^*\tilde{K}^{1/2}y \leq \sqrt{y^*yy^*\tilde{K}y} \tag{25}$$

Substituting Eq. (25) into Eq. (22) and rearranging terms yields

$$(y^*\tilde{C}y/2\sqrt{y^*yy^*\tilde{K}y}) \leq a \tag{26}$$

which is valid for all nonzero complex n-dimensional vectors y. In particular, if y is set to s_j, then the left-hand side of Eq. (26) is equivalent to Eq. (19), or

$$\zeta_j \leq a$$

for all $j = 1,\ldots,n$. Therefore, if the matrix $(2a\tilde{K}^{1/2} - \tilde{C})$ is positive semidefinite, then all modal damping ratios $\zeta_j \leq a$ where $a \geq 0$.

Next, two alternative upper bound modal damping ratio criteria are derived. The derivation is very similar to the underdamped condition derivation presented in the second and third paragraphs of Sec. IIA. Since the matrices $4a^2\tilde{K}$ and $\tilde{C}$ are positive semidefinite, then, according to Bellman,[4] if $(4a^2\tilde{K} - \tilde{C}^2)$ is positive semidefinite, so is $(2a\tilde{K}^{1/2} - \tilde{C})$. Consequently, if the matrix $(4a^2\tilde{K} - \tilde{C}^2)$ is positive semidefinite, then all modal damping ratios $\zeta_j \leq a$, where $a \geq 0$. This, the second upper bound modal damping ratio criterion, avoids computation of the square root of $\tilde{K}$; therefore, this criterion is simpler to evaluate than the previously stated criterion.

The third upper bound modal damping ratio criterion is derived from the second. Using the argument stated in the derivation of the third underdamped condition presented in the third paragraph of Sec. IIA, with $4\tilde{K}$ replaced with $4a^2\tilde{K}$, yields that the matrix $(4a^2K - CM^{-1}C)$ is positive semidefinite if and only if the matrix $(4a^2\tilde{K} - \tilde{C}^2)$ is positive semidefinite. Therefore, if the matrix $(4a^2K - CM^{-1}C)$ is positive semidefinite, then all modal damping ratios $\zeta_j \leq a$, where $a \geq 0$. Note that since the evaluation of the square root of M is avoided, then this, the third upper bound modal damping ratio criterion, requires less computation than the second or first criteria.

The three upper bound modal damping ratio criteria just derived are combined into a single theorem:

Theorem 1

If an MDOF system is stable (M and K are positive definite and C is positive semidefinite), underdamped ($2\tilde{K}^{1/2} - \tilde{C}$, $4\tilde{K} - \tilde{C}^2$, or $4K - CM^{-1}C$ is positive definite), and at least one of the matrices $(2a\tilde{K}^{1/2} - \tilde{C})$, $(4a^2\tilde{K} - \tilde{C}^2)$, or $(4a^2K - CM^{-1}C)$ is positive semidefinite, then all modal damping ratios $\zeta_j \leq a$, where $a \geq 0$.

C. "Constant" Modal Damping Ratio Criteria

Here, three constant modal damping ratio criteria are presented. First, it is shown that if $2a\tilde{K}^{1/2} = \tilde{C}$, then all $\zeta_j = a$, where $a \geq 0$. If this statement holds, then the scalar a is equivalent to all of the system modal damping ratios. Afterwards, two alternate constant modal damping ratio criteria are derived.

Suppose that $2a\tilde{K}^{1/2} = \tilde{C}$, where a is a real non-negative constant scalar. The constant modal damping ratio test condition is equivalent to

$$2ay^*\tilde{K}^{1/2}y = y^*\tilde{C}y \tag{27}$$

where y is any complex n-dimensional vector. If y is set to s_j, one of the jth transformed system eigenvectors, then Eq. (27) becomes

$$2as_j^*\tilde{K}^{1/2}s_j = s_j^*\tilde{C}s_j \tag{28}$$

Since in this case $\tilde{K}^{1/2}$ is proportional to $\tilde{C}$, then the MDOF system given by Eq. (1) has normal modes; therefore, according to Caughey and O'Kelly,[8] s_j is also an eigenvector of $\tilde{C}$, $\tilde{K}$, and $\tilde{K}^{1/2}$. Equation (24) (obtained from the Cauchy-Schwarz inequality) is restated here as

$$(y^*\tilde{K}^{1/2}y)^2 \leq y^*yy^*\tilde{K}y \tag{29}$$

If $y = s_j$, an eigenvector of $\tilde{K}$ in this case, then Eq. (29) is an equality,[9] or

$$(s_j^*\tilde{K}^{1/2}s_j)^2 = s_j^*s_js_j^*\tilde{K}s_j \tag{30}$$

Since the matrix $\tilde{K}$ is positive definite, then both sides of Eq. (30) are positive. Taking the positive square root of both sides of Eq. (30), substituting into Eq. (28), and rearranging terms yields

$$(s_j^*\tilde{C}s_j/2\sqrt{s_j^*s_js_j^*\tilde{K}s_j}) = a \tag{31}$$

The left-hand side of Eq. (31) is equivalent to Eq. (19), or

$$\zeta_j = a$$

for all $j = 1,\ldots,n$. Therefore, if $2a\tilde{K}^{1/2} = \tilde{C}$, then all modal damping ratios are constant, i.e., $\zeta_j = a$, where $a \geq 0$.

Next, two alternate constant modal damping ratio criteria are determined. Since the matrices $a\tilde{K}$ and $\tilde{C}$ are positive semidefinite, then $2a\tilde{K}^{1/2} = \tilde{C}$ is equivalent to $4a^2\tilde{K} = \tilde{C}^2$. Therefore, $4a^2\tilde{K} = \tilde{C}^2$, then all modal damping ratios $\zeta_j = a$, where $a \geq 0$. Substituting $\tilde{C} = M^{-1/2}CM^{-1/2}$ and $\tilde{K} = M^{1/2}KM^{-1/2}$ into $4a^2\tilde{K} = \tilde{C}^2$, and premultiplying and postmultiplying by $M^{1/2}$ yields the equivalent expression, $4a^2K = CM^{-1}C$. Consequently, if $4a^2K = CM^{-1}C$, then all modal damping ratios $\zeta_j = a$, where $a \geq 0$. Comparing this, the third, with the first and second constant modal damping ratio criteria, the third criterion is easiest to evaluate since computation of the square root of the matrices M or $\tilde{K}$ is avoided.

The three constant modal damping ratio criteria just presented can be combined into a theorem:

Theorem 2

If an MDOF system is stable, underdamped, and at least one of the equalities $2a\tilde{K}^{1/2} = \tilde{C}$, $4a^2\tilde{K} = \tilde{C}^2$, or $4a^2K = CM^{-1}C$ holds, then all modal damping ratios $\zeta_j = a$, where $a \geq 0$.

D. "Lower Bound" Modal Damping Ratio Criteria (Normal Mode Case)

In this section, three normal mode lower bound modal damping ratio criteria are derived. First, it is proved that if $(\tilde{C} - 2a\tilde{K}^{1/2})$ is positive semidefinite and the MDOF system has normal modes,[8] then all $\zeta_j \geq a$, where $a \geq 0$. If this statement holds, then the scalar a provides a lower bound on all of the system modal damping ratios. Finally, two alternate normal mode lower bound modal damping ratio criteria are derived.

By hypothesis, the matrix $(\tilde{C} - 2a\tilde{K}^{1/2})$ is positive semidefinite, where a is a real non-negative constant scalar. Since the matrix $(\tilde{C} - 2a\tilde{K}^{1/2})$ is real and symmetric, then

$$y^*(\tilde{C} - 2a\tilde{K}^{1/2})y \geq 0 \tag{32}$$

for all complex n-dimensional vectors y (Ref. 5). If y is set to s_j, one of the jth transformed system eigenvectors, then Eq. (32) becomes

$$s_j^*\tilde{C}s_j - 2as_j^*\tilde{K}^{1/2}s_j \geq 0 \tag{33}$$

According to Caughey and O'Kelly,[8] if Eq. (1) is a normal mode MDOF system, then s_j is also an eigenvector of $\tilde{C}$, $\tilde{K}$, and $\tilde{K}^{1/2}$; therefore, Eq. (30) is valid, or

$$(s_j^*\tilde{K}^{1/2}s_j)^2 = s_j^*s_js_j^*\tilde{K}s_j \tag{34}$$

Since the matrix $\tilde{K}$ is positive definite, then both sides of Eq. (34) are positive. Taking the positive square root of both sides of Eq. (34), substituting into Eq. (33), and rearranging terms yields

$$(s_j^*\tilde{C}s_j/2\sqrt{s_j^*s_js_j^*\tilde{K}s_j}) \geq a \tag{35}$$

The left-hand side of Eq. (35) is equivalent to Eq. (19), or

$$\zeta_j \geq a$$

for all $j = 1,\ldots,n$. Therefore, if the matrix $(\tilde{C} - 2a\tilde{K}^{1/2})$ is positive semidefinite and the MDOF system given by Eq. (1) has normal modes, then all $\zeta_j \geq a$, where $a \geq 0$.

Two alternate normal mode lower bound modal damping ratio criteria are now derived. The derivation is very similar to the underdamped condition derivation presented in the second and third paragraphs of Sec. IIA. According to Bellman,[4] since the matrices $\tilde{C}$ and $4a^2\tilde{K}$ are positive semidefinite, then, if $(\tilde{C}^2 - 4a^2\tilde{K})$ is positive semidefinite, so is $(\tilde{C} - 2a\tilde{K}^{1/2})$. Therefore, if the matrix $(\tilde{C}^2 - 4a^2\tilde{K})$ is positive semidefinite and the system has normal modes, then all $\zeta_j \geq a$, where $a \geq 0$. This, the second normal mode lower bound modal damping ratio criterion, avoids computation of the square root of $\tilde{K}$; therefore, this criterion is simpler to evaluate than the previously stated criterion.

The third normal mode lower bound modal damping ratio criterion is derived from the second. Using the argument stated in the derivation of the third underdamped condition presented in the third paragraph of Sec. IIA, with $\tilde{C}^2$ and $4\tilde{K}$ switched, and $4\tilde{K}$ replaced with $4a^2\tilde{K}$, yields that the matrix $(CM^{-1}C - 4a^2K)$ is positive semidefinite if and only if the matrix $(\tilde{C}^2 - 4a^2\tilde{K})$ is positive semidefinite. Therefore, if the matrix $(CM^{-1}C - 4a^2K)$ is positive semidefinite and the system has normal modes then all $\zeta_j \leq a$ where $a \geq 0$. Since the evaluation of the square root of M is avoided, then this, the third normal mode lower bound modal damping ratio criterion, requires less computation than the first or second criteria.

The following theorem combines the three normal mode lower bound modal damping criteria just derived.

Theorem 3

If an MDOF system is stable, underdamped, has normal modes, and at least one of the matrices $(\tilde{C} - 2a\tilde{K}^{1/2})$, $(\tilde{C}^2 - 4a^2\tilde{K})$, or $(CM^{-1}C - 4a^2K)$ is positive semidefinite, then all modal damping ratios $\zeta_j \geq a$, where $a \geq 0$.

E. "Lower Bound" Modal Damping Ratio Criteria (General Case)

Here, four general lower bound modal damping ratio criteria are presented. First, it is shown that if $(\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I)$ is positive semidefinite, then all $\zeta_j \geq a$, where $\epsilon_1 = 2a[\sqrt{\lambda_1(\tilde{K})} - \sqrt{\lambda_n(\tilde{K})}]$ and $a \geq 0$. The quantities $\lambda_1(\cdot)$ and $\lambda_n(\cdot)$ denote the maximum and minimum eigenvalues, respectively. The quantity I is the $n \times n$ identity matrix. Next, it is proved that if $(\tilde{C}^2 - 4a^2\tilde{K} - \epsilon_2 I)$ is positive semidefinite, then all $\zeta_j \geq a$, where $\epsilon_2 = [\lambda_1(\tilde{C})]^2 - [\lambda_n(\tilde{C})]^2$ and $a \geq 0$. Lastly, two lower bound damping ratio criteria are derived that relate directly to the MDOF damping ratio expression, Eq. (19).

By hypothesis, the matrix $(\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I)$ is positive semidefinite where $\epsilon_1 = 2a[\sqrt{\lambda_1(\tilde{K})} - \sqrt{\lambda_n(\tilde{K})}]$, $\lambda_1(\cdot)$ and $\lambda_n(\cdot)$ denote the maximum and minimum eigenvalue, respectively, and a is a real non-negative constant scalar. Since the matrix $(\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I)$ is real and symmetric, then

$$y^*(\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I)y \geq 0 \tag{36}$$

for all complex n-dimensional vectors y (Ref. 5). Expanding Eq. (36) yields

$$y^*\tilde{C}y - 2ay^*\tilde{K}^{1/2}y - \epsilon_1 y^*y \geq 0 \tag{37}$$

where $y^*y = y^*Iy$.

The next step in the proof is to reduce the expression for ϵ_1 to a more usable form. The Rayleigh inequality is given by

$$\lambda_n(A) \leq (q^*Aq/q^*q) \leq \lambda_1(A) \tag{38}$$

where A is a real $n \times n$ symmetric matrix and q is an arbitrary nonzero complex n-dimensional vector.[1,5] Substituting $A = \tilde{K}$ and $q = y$ into Eq. (38) results in

$$\lambda_1(\tilde{K}) \geq (y^*\tilde{K}y/y^*y) \tag{39}$$

Since $\tilde{K}$ is positive definite, then both sides of Eq. (39) are positive; therefore, taking the positive square root of both sides of Eq. (39) will not change the nature of the inequality, or

$$\sqrt{\lambda_1(\tilde{K})} \geq \sqrt{(y^*\tilde{K}y/y^*y)} \tag{40}$$

Substituting $A = \tilde{K}^{1/2}$ and $q = y$ into Eq. (38) results in

$$\lambda_n(\tilde{K}^{1/2}) \leq (y^*\tilde{K}^{1/2}y/y^*y) \tag{41}$$

Since the matrices $\tilde{K}$ and $\tilde{K}^{1/2}$ are positive definite and have the same eigenvectors, then

$$\sqrt{\lambda_1(\tilde{K})} = \lambda_i(\tilde{K}^{1/2}) \tag{42}$$

where $\lambda_i(\cdot)$ denotes the ith eigenvalue. Applying Eq. (42) to Eq. (41) results in

$$\sqrt{\lambda_n(\tilde{K})} \leq (y^*\tilde{K}^{1/2}y/y^*y) \tag{43}$$

Substituting Eqs. (40) and (43) into the expression for $\epsilon_1 = 2a[\sqrt{\lambda_1(\tilde{K})} - \sqrt{\lambda_n(\tilde{K})}]$ yields

$$\epsilon_1 \geq 2a\sqrt{(y^*\tilde{K}y/y^*y)} - 2a[(y^*\tilde{K}^{1/2}y/y^*y)] \tag{44}$$

Now, Eq. (37) is reduced to the form of Eq. (19). Substituting Eq. (44) into Eq. (37) results in

$$y^*\tilde{C}y - 2ay^*\tilde{K}^{1/2}y - 2a\sqrt{y^*yy^*\tilde{K}y} - 2ay^*\tilde{K}^{1/2}y \geq 0$$

Simplifying and rearranging terms yields

$$(y^*\tilde{C}y/2\sqrt{y^*yy^*\tilde{K}y}) \geq a \tag{45}$$

which is valid for all nonzero n-dimensional vectors y. In particular, if y is set to s_j, then the left-hand side of Eq. (45)

is equivalent to Eq. (19), or

$$\zeta_j \geq a$$

for all $j = 1, \ldots, n$. Therefore, if the matrix $(\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I)$ is positive semidefinite then all $\zeta_j \geq a$, where $\epsilon_1 = 2a[\sqrt{\lambda_1(\tilde{K})} - \sqrt{\lambda_n(\tilde{K})}]$ and $a \geq 0$. The next general lower bound modal damping ratio criterion has a similar proof.

Suppose that the matrix $(\tilde{C}^2 - 4a^2\tilde{K} - \epsilon_2 I)$ is positive semidefinite, where $\epsilon_2 = [\lambda_1(\tilde{C})]^2 - [\lambda_n(\tilde{C})]^2$ and a is a real non-negative constant scalar. Since the matrix $(\tilde{C}^2 - 4a^2\tilde{K} - \epsilon_2 I)$ is real and symmetric, then

$$y^*(\tilde{C}^2 - 4a^2\tilde{K} - \epsilon_2 I)y \geq 0 \tag{46}$$

for all complex n-dimensional vectors y (Ref. 5). Expanding Eq. (46) yields

$$y^*\tilde{C}^2 y - 4a^2 y^*\tilde{K}y - \epsilon_2 y^* y \geq 0 \tag{47}$$

Next, the expression for ϵ_2 is reduced to a more usable form. Substituting $A = \tilde{C}^2$ and $q = y$ into the Rayleigh inequality, Eq. (38), results in

$$\lambda_1(\tilde{C}^2) \geq (y^*\tilde{C}^2 y / y^* y) \tag{48}$$

Since the matrices $\tilde{C}$ and $\tilde{C}^2$ are positive semidefinite and have the same eigenvectors, then

$$[\lambda_1(\tilde{C})]^2 = \lambda_i(\tilde{C}^2) \tag{49}$$

where $\lambda_i(\cdot)$ denotes the ith eigenvalue. Applying Eq. (49) to Eq. (48) results in

$$[\lambda_1(\tilde{C})]^2 \geq (y^*\tilde{C}^2 y / y^* y) \tag{50}$$

Substituting $A = \tilde{C}$ and $q = y$ into Eq. (38) yields

$$\lambda_n(\tilde{C}) \leq (y^*\tilde{C}y / y^* y) \tag{51}$$

Since $\tilde{C}$ is positive semidefinite, then both sides of Eq. (51) are non-negative; therefore, squaring both sides of Eq. (51) will not change the nature of the inequality, or

$$[\lambda_n(\tilde{C})]^2 \leq [(y^*\tilde{C}y / y^* y)]^2 \tag{52}$$

Substituting Eqs. (50) and (52) into the expression for $\epsilon_2 = [\lambda_1(\tilde{C})]^2 - [\lambda_n(\tilde{C})]^2$ yields

$$\epsilon_2 \geq (y^*\tilde{C}^2 y / y^* y) - [(y^*\tilde{C}y / y^* y)]^2 \tag{53}$$

Now, Eq. (47) is reduced to the form of Eq. (19). Substituting Eq. (53) into Eq. (47) results in

$$y^*\tilde{C}^2 y - 4a^2 y^*\tilde{K}y - y^*\tilde{C}^2 y + [(y^*\tilde{C}y)^2 / y^* y] \geq 0$$

Simplifying and rearranging terms yields

$$[(y^*\tilde{C}y)^2 / 4y^* y y^*\tilde{K}y] \geq a^2 \tag{54}$$

Since both sides of Eq. (54) are non-negative, taking the positive square root will not change the nature of the inequality, or

$$(y^*\tilde{C}y / 2\sqrt{y^* y y^*\tilde{K}y}) \geq a \tag{55}$$

which is valid for all nonzero n-dimensional vectors y. In particular, if y is set to s_j, then the left-hand side of Eq. (55) is equivalent to Eq. (19), or

$$\zeta_j \geq a$$

for all $j = 1, \ldots, n$. Therefore, if the matrix $(\tilde{C}^2 - 4a^2\tilde{K} - \epsilon_2 I)$ is positive semidefinite, then all modal damping ratios $\zeta_j \geq a$, where $\epsilon_2 = [\lambda_1(\tilde{C})]^2 - [\lambda_n(\tilde{C})]^2$ and $a \geq 0$. This is the second general lower bound modal damping ratio criterion.

Now, suppose that the scalar

$$\lambda_n(\tilde{C}) - 2a\sqrt{\lambda_1(\tilde{K})} \geq 0 \tag{56}$$

where a is a real non-negative constant scalar. Substituting Eqs. (40) and (51) into Eq. (56) yields

$$(y^*\tilde{C}y / y^* y) - 2a\sqrt{(y^*\tilde{K}y / y^* y)} \geq 0 \tag{57}$$

which is valid for all nonzero n-dimensional vectors y. In particular, if y is set to s_j, then Eq. (57) becomes

$$(s_j^*\tilde{C}s_j / 2\sqrt{s_j^* s_j s_j^*\tilde{K}s_j}) \geq a \tag{58}$$

The left-hand side of Eq. (58) is equivalent to Eq. (19), or

$$\zeta_j \geq a$$

for all $j = 1, \ldots, n$. Therefore, if the scalar $\lambda_n(\tilde{C}) - 2a\sqrt{\lambda_1(\tilde{K})}$ is non-negative, then all $\zeta_j \geq a$, where $a \geq 0$.

Since $\lambda_n(\tilde{C})$ and $2a[\lambda_1(\tilde{K})]^{1/2}$ are non-negative, then Eq. (56) is equivalent to

$$[\lambda_n(\tilde{C})]^2 - 4a^2\lambda_1(\tilde{K}) \geq 0$$

Therefore, if the scalar $[\lambda_n(\tilde{C})]^2 - 4a^2\lambda_1(\tilde{K})$ is non-negative, then all $\zeta_j \geq a$, where $a \geq 0$.

The theorem that combines the four general lower bound modal damping criteria just presented is as follows.

Theorem 4

If an MDOF system is stable, underdamped, and at least one of the matrices $(\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I)$ or $(\tilde{C}^2 - 4a^2\tilde{K} - \epsilon_2 I)$ is positive semidefinite, or one of the scalars $\{\lambda_n(\tilde{C}) - 2a[\lambda_1(\tilde{K})]^{1/2}\}$ or $\{[\lambda_n(\tilde{C})]^2 - 4a^2\lambda_1(\tilde{K})\}$ is non-negative, then all modal damping ratios $\zeta_j \geq a$, where $\epsilon_1 = 2a\{[\lambda_1(\tilde{K})]^{1/2} - [\lambda_n(\tilde{K})]^{1/2}\}$, $\epsilon_2 = [\lambda_1(\tilde{C})]^2 - [\lambda_n(\tilde{C})]^2$, and $a \geq 0$.

III. Design Example

In the following design example, one of the possible uses of Theorem 1 (upper bound modal damping ratio criterion) and Theorem 4 (general lower bound modal damping ratio criterion) is illustrated. Consider a three degree-of-freedom system of the form of Eq. (1) with

$$M = I, C = \begin{bmatrix} c_1 & 0.3 & 0 \\ 0.3 & 0.8 & 0.1 \\ 0 & 0.1 & 0.8 \end{bmatrix}, \quad K = \begin{bmatrix} 2.2 & 0.7 & -0.6 \\ 0.7 & 1.6 & 0 \\ -0.6 & 0 & 2.2 \end{bmatrix}$$

where I is the 3×3 identity matrix and c_1 is a constant scalar. Here, the scalar c_1 is the only element in the matrices M, C, and K that is permitted to vary. Note that M, C, and K are symmetric. In addition, this is a non-normal mode system since $CM^{-1}K \neq KM^{-1}C$ (Ref. 8). The following units are assumed unless specified otherwise: mass is given in kilograms, damping in kilogram/second, stiffness in Newton/meter, time in seconds, displacement in meters, and force in Newtons. The design objective is to determine the minimum and maximum values of c_1 such that all modal damping ratios lie between 0.1 and 0.4 and the underdamped and stability conditions are satisfied.

The stability definiteness condition is satisfied if the matrices M and K are positive definite and the matrix C is positive semidefinite. As given, the matrices M and K are positive definite. The matrix C is positive semidefinite if all of the determinants of its principal minors are non-negative,[1] or if

$$c_1 \geq 0.1143$$

Therefore, the system is stable if $c_1 \geq 0.1143$.

The underdamped definiteness condition is satisfied if the matrix $(2\tilde{K}^{1/2} - \tilde{C})$ is positive definite. Since the mass matrix is the identity matrix, $\tilde{K} = K$ and $\tilde{C} = C$; therefore,

$$2\tilde{K}^{1/2} - \tilde{C} = \begin{bmatrix} 2.8899 - c_1 & 0.2251 & -0.4155 \\ 0.2251 & 1.6744 & -0.0597 \\ -0.4155 & -0.0597 & 2.1370 \end{bmatrix}$$

The matrix $(2\tilde{K}^{1/2} - \tilde{C})$ is positive definite if all of the determinants of its principal minors are positive,[1] or if

$$c_1 < 2.7818$$

Therefore, the system is underdamped if $c_1 < 2.7818$.

To achieve the modal damping ratio part of the design objective, Theorems 1 and 4 are used. Theorem 4 (general lower bound damping ratio criterion) is favored over Theorem 3 (normal mode lower bound damping ratio criterion) since this is a non-normal mode system. The stability and underdamped requirements considered in the previous two paragraphs are contained in Theorems 1 and 4. Therefore, when testing the damping ratio conditions within Theorems 1 and 4, the permissible values of c_1 lie in the range $0.1143 \leq c_1 < 2.7818$.

All modal damping ratios lie above 0.1 if the matrix $(\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I)$ in Theorem 4 is positive semidefinite, with $a = 0.1$. The matrix

$$\tilde{C} - 2a\tilde{K}^{1/2} - \epsilon_1 I = \begin{bmatrix} c_1 - 0.1457 & 0.2475 & 0.0416 \\ 0.2475 & 0.6959 & 0.0960 \\ 0.0416 & 0.0960 & 0.6496 \end{bmatrix}$$

is positive semidefinite if all of the determinants of the principal minors are non-negative,[1] or if

$$c_1 \geq 0.2337$$

Therefore, all modal damping ratios lie above 0.1 if $c_1 \geq 0.2337$. Verifying this result, the exact damping ratios of this system with $c_1 = 0.2337$ are $\zeta_j = 0.1786$, 0.1357, and 0.3387, which lie above 0.1.

All modal damping ratios lie below 0.4 if the matrix $(2a\tilde{K}^{1/2} - \tilde{C})$ in Theorem 1 is positive semidefinite, with $a = 0.4$. The matrix

$$2a\tilde{K}^{1/2} - \tilde{C} = \begin{bmatrix} 1.1560 - c_1 & -0.0900 & -0.1662 \\ -0.0900 & 0.1898 & -0.0839 \\ -0.1662 & -0.0839 & 0.3748 \end{bmatrix}$$

is positive semidefinite if all of the determinants of the principal minors are non-negative,[1] or if

$$c_1 \leq 0.9870$$

Therefore, all modal damping ratios lie below 0.4 if $c_1 \leq 0.9870$. Verifying this result, the exact damping ratios of this system with $c_1 = 0.9870$ are $\zeta_j = 0.2896$, 0.3661, and 0.2736, which lie below 0.4. In summary, for the range of values for c_1 given by

$$0.2337 \leq c_1 \leq 0.9870$$

the system is underdamped, stable, and all modal damping ratios lie between 0.1 and 0.4.

The design procedure is summarized as follows. First, a design objective is stated. In order that the modal damping ratio design criteria remain valid, the design objective must include constraints that insure that the system is stable and underdamped. In the present example, the design objective was to determine the maximum range of values that a single element of the damping matrix could attain such that the system modal damping ratios remained in a specified range. Other similar design objectives may allow more elements of the system matrices to vary, and/or have a different modal damping ratio constraint. The design objective in this example was kept simple in order to illustrate the design methodology.

Second, a solution to the design objective is determined. In the present example, a solution was obtained by solving a set of simple inequalities. For intricate design objectives, however, finding a solution may require solving a set of nonlinear algebraic inequalities. Rather than using the modal damping ratio design criteria to determine a solution to the design objective, it is possible to examine the exact modal damping ratios directly. In this case, however, determining a solution to the design objective will likely require a trial-and-error or constrained optimization approach: several constrained optimization routines are described in Reklaitis et al.[10] In general, these constrained optimization routines require orders of magnitude more computational effort than that required to solve a set of nonlinear inequalities. Therefore, to determine a solution to the design objective requires less computational effort if the modal damping ratio criteria are used rather than the exact modal damping ratios.

Third and last, the solution is verified. For problems of large dimension this step might not be practical. In the design example, the solution was verified by examining the actual system modal damping ratios.

It is important to note that a three degree-of-freedom example was chosen for simplicity of presentation; the design procedure is not limited to systems of low order. Although it is true in general that most design methods work best for low-order models, the method presented here works equally well for high- and low-order models. The choice of the design objective may cause some eventual limitation of order.

IV. Conclusions

Simple design criteria were presented that provide bounds on the values of the modal damping ratios of an underdamped, stable linear-lumped parameter system. First, a general expression for the system modal damping ratios was derived. This expression is valid for stable, underdamped systems. Using the modal damping ratio expression as a focus, four theorems were developed. The first and second theorems are, respectively, criteria for determining whether all of the system modal damping ratios are less than or equal to a given value. Similarly, the third and fourth theorems are criteria for determining whether all of the system modal damping ratios are greater than a given value. The third theorem applies exclusively to the normal mode case. The fourth theorem applies to the general normal and non-normal mode cases. An example illustrated how the modal damping ratio criteria might be used in design.

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